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APPROXIMATE FORMULAS FOR THE HEAT FLUXES TO THE SURFACE OF THREE-DIMENSIONAL BODIES†

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The steady three-dimensional flow of a perfect gas over blunt bodies is investigated using a model of a hypersonic viscous shock layer. Analytical formulas are obtained for the distribution of the heat flux on the side surface with respect to its value at the stagnation point using the integral method of successive approximations. It is shown that at moderate and high Reynolds numbers the distribution of the relative heat flux depends slightly on the Reynolds number and on other gas-dynamic parameters of the flow. The accuracy of the formulas obtained is estimated by comparing them with the results of numerical solutions of the three-dimensional viscous shock layer equations for bodies of various shape. Similar formulas are obtained in [1] for the neighbourhood of a plane of symmetry of three-dimensional bodies. Note that, unlike the formulas for the relative heat fluxes assumed in boundary-layer theory (for example, [2, 3]), the formulas proposed here do not require values of the flow parameters on the outer edge of the boundary layer (i.e. the calculation of three-dimensional inviscid flow) and a calculation of the inviscid streamlines on the surface. They depend only on the geometrical characteristics of the body.

SUPPOSE the surface of the body is specified in a Cartesian system of coordinates by the equation $z=f(x, y)$, the vector of the free stream velocity V_∞ coincides in direction with the z axis, the origin of coordinates is placed at the stagnation point of the flow, and the x and y axes are situated in the planes of principal curvature of the surface at this point. We will choose a system of curvilinear non-orthogonal coordinates $\{x^i\}$ normal to the surface; x^3 is the distance along the normal to the surface, and, as the two others, chosen on the surface, we will use Cartesian coordinates of the point of intersection of this normal with the surface: $x^1=x, x^2=y, z=f(x^1, x^2)$. For the equations of the three-dimensional hypersonic viscous shock layer in the system of coordinates $\{x^i\}$ [4] numerical calculations showed that for $\mathrm{Re} \geqslant 100$, to determine the heat flux, with respect to its value at the stagnation point, one can use as the boundary conditions the conditions of adhesion on the body and the usual Rankine–Hugoniot relations. The distributions of the relative heat flux over the body surface will be the same as when calculating the effect of slipping on the body surface using the modified Rankine–Hugoniot relations at the shock.

When the equations for the control functions were solved in the locally self-similar approximation, a simple formula was obtained [5] for the relative heat flux in the first approximation of the method of successive approximations [6]. In [5] the initial approximation for the pressure was specified by Newton's formula, which in some cases may lead to considerable errors in determining the heat flux. If we take into account the difference between the pressure distribution and the Newtonian distribution, i.e. we determine the initial approximation for the pressure from the momentum equation in a projection on to the normal to the surface, to a first approximation of the method of successive approximations we can obtain the following formula for the relative heat flux:

$$\frac{q}{q_0}=\cos^{3/2}\alpha\sqrt{\frac{H}{H_0\lambda}}, \qquad \lambda=1+\frac{4}{15}\,\frac{f_{xx}f_x^2+2f_{xy}f_xf_y+f_{yy}f_y^2}{\cos^3\alpha H} \tag{1}$$

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Here H is the mean curvature of the surface at the point considered, equal to the half-sum of the principal curvatures of the surface at this point:

$$H=\frac{f_{xx}(1+f_y)+f_{yy}(1+f_x)-2f_{xy}f_xf_y}{2[1+f_x^2+f_y^2]^{3/2}} \tag{2}$$

$$\cos\alpha=(1+f_x^2+f_y^2)^{-1/2}$$

The zero subscript corresponds to the stagnation point of the flow, the factor λ characterizes the difference between the pressure distribution and the Newtonian distribution and α is the angle between the direction of the free stream velocity and the vector of the normal to the surface.

In view of the fact that when deriving (1) a locally self-similar approximation was employed, it may turn out to be insufficiently accurate for certain bodies.

More accurate results can be obtained by using a combined approach, which consists of using the axisymmetrical analogy [7] and the formula for the relative heat flux on the surface of an axisymmetrical body [1], obtained without using the locally self-similar approximation.

A similarity relation was established in [7], which expresses the heat flux on the side surface of a three-dimensional body in terms of its value on the surface of a certain equivalent axisymmetrical body. The shape of the latter is determined solely by the geometrical characteristics and is independent of the gas-dynamic parameters of the flow. At moderate and high Reynolds numbers (Re $\geqslant$ 100), this similarity relation can be written as follows.

Suppose the surface of the body is specified in a cylindrical system of coordinates r, z, φ by the equation $r=r(z,\varphi)$, where the z axis passes through the stagnation point and is directed along the vector of the free stream velocity V_∞. Then, the heat flux q on the surface of the three-dimensional body along a fixed meridional section $\varphi=\varphi_*$ is given by the relation

$$q=\sqrt{H/H^0}\,q^0 \tag{3}$$

Here q^0 is the heat flux to the surface of the equivalent axisymmetrical body, the shape of which is given by the relation

$$r^0(z)=\int_0^z r_z(1+(r_\varphi/r)^2)^{-1/2}dz\,|_{\varphi=\varphi^*} \tag{4}$$

where H is the mean curvature of the surface of the three-dimensional body at the point considered and H^0 is the mean curvature of the surface of the equivalent axisymmetrical body at the given point.

The equivalent axisymmetrical body was constructed in such a way that the angle between the normal to its generatrix and the direction of the free stream velocity α^0 varied along the generatrix in the same way as the angle α (between the normal to the surface of the three-dimensional body and the velocity V_∞) along the chosen meridional section, i.e. $\alpha^0=\alpha$.

In a cylindrical system of coordinates we have

$$H=\frac{r_{zz}r(r^2+r_\varphi^2)+(1+r_z^2)(r_{\varphi\varphi}r-2r_\varphi^2-r^2)-2r_zr_\varphi(r_{z\varphi}r-r_zr_\varphi)}{2[r^2(1+r_z^2)+r_\varphi^2]^{3/2}} \tag{5}$$

$$H^0=\frac{r^0(r^0)_{zz}-(1+(r_z^0)^2)}{2r^0(1+(r_z^0)^2)^{3/2}}$$

At the stagnation point, relation (3) takes the form

$$q_0=\sqrt{\frac{1+k}{2}}\,q_0^0 \tag{6}$$

Here k is the ratio of the principal curvatures of the surface at the stagnation point and q_0^0 is the heat flux at the stagnation point of the axisymmetrical body, for example, a sphere.

The following approximate formula was obtained in [1] for the distribution of the relative heat flux along the surface of an axisymmetrical body:

$$\frac{q^0}{q_0^0} = \frac{\cos^2\alpha \sin\alpha r^0}{2I}, \quad I = \lambda^0 \left[\int_0^s \frac{\cos^2\alpha \sin\alpha {r^0}^2}{\lambda^0} ds\right]^{1/2} \tag{7}$$
$$\lambda^0 = 1 + \frac{4}{15}\frac{\mathrm{tg}^2\alpha}{R^0 H^0}, \quad R^0 = \frac{(1+(r^0)_z^2)^{3/2}}{(r^0)_{zz}}$$

Here s is the length of the arc along the generatrix, measured from the stagnation point, and R_0 is the radius of curvature of the generatrix.

Taking relations (3), (6) and (7) into account we obtain the following relation for the value of the relative heat flux along the meridional section of the three-dimensional body

$$\frac{q}{q_0} = \frac{H^{1/2}\cos^2\alpha \sin\alpha r^0}{[2(1+k)H^0]^{1/2} I} \tag{8}$$

This formula has an invariant form and is independent of the choice of the system of coordinates. If the surface of the body is specified in a cylindrical system of coordinates, the value of r^0, which specifies the form of the equivalent axisymmetrical body, is found from (4) while the quantities H, H^0 and λ^0 are found from (5) and (7). If the surface of the body is specified in a Cartesian system of coordinates x, y, z, then, for the radius r^0 of the equivalent axisymmetrical body corresponding to the chosen meridional section $y = cx$, we obtain the following parametric expression:

$$r^0(x) = \int_0^x [f_x(x, cx) + cf_y(x, cx)][f_x^2(x, cx) + f_y^2(x, cx)]^{-1/2} dx \tag{9}$$
$$z(x) = f(x, cx)$$

In a Cartesian system of coordinates H is found from (2) and H^0 and λ^0 are found from (5) and (7) where we must put

$$r_z^0 = (f_x^2 + f_y^2)^{-1/2}$$
$$r_{zz}^0 = \frac{f_x(f_{xx} + cf_{yy}) + f_y(f_{xy} + cf_{yy})}{(f_x^2 + f_y^2)^{3/2}(f_x + cf_y)}$$

Formula (8) assumes integration along the surface of the equivalent axisymmetrical body, the form of which is not known in advance and which must be calculated initially. When carrying out the calculations, it is more convenient to carry out the integration over a previously known coordinate. Taking into account the fact that $\alpha = \alpha^0$, and changing from integration over s to integration over z, we can obtain a form of (8) that is more convenient for practical use, namely,

$$\frac{q}{q_0} = \frac{1}{\lambda^0} H^{1/2}\cos^2\alpha \sin\alpha r^0 \left[2(1+k)H^0 \int_0^z \frac{\cos^2\alpha (r^0)^2}{\lambda^0} dz\right]^{-1/2} \tag{10}$$

It follows from the approximate relations obtained for q/q_0 that for medium and high Reynolds numbers ($\mathrm{Re} \gtrsim 100$) the value of the relative heat flux on the side surface of the body ceases to depend on the Re number, (this relationship is important at low Re numbers), and also on the surface temperature T_w and the ratio of the specific heat capacities γ and is determined solely by the geometrical properties of the surface.

This is confirmed by numerical calculations of the system of governing equations. In Fig. 1 we show the results of calculations of the relative heat flux on the side surface of an elliptic paraboloid with $k = 0.4$ and a hyperboloid with a semi-aperture angle of 40° in the $y = 0$ plane and $k = 0.5$. Strips 1 and 2 contain all curves of the distributions q/q_0 along the meridian $\varphi = 45°$ on the surface of the paraboloid and the hyperboloid when the Re number varies from 10^2–5×10^4, T_w varies from 0.01 to 0.25 and γ varies from 1.15 to 1.667. Along the abscissa axis we have plotted $r = \sqrt{x^2 + y^2}$ —the distance from the point on the surface considered to the z axis.

To estimate the accuracy of (1) and (8) we made a systematic comparison of the calculations with the results of a numerical solution of the system of equations of a three-dimensional thin viscous shock layer. We used a finite-difference method with fourth-order accuracy of approximation in a transverse direction and second-

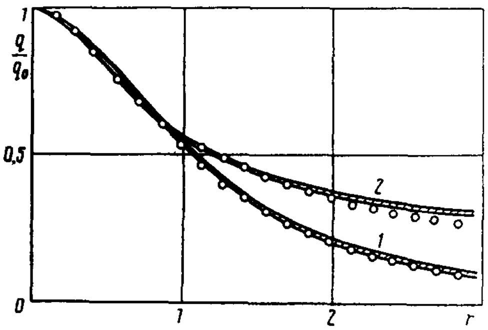

FIG. 1.

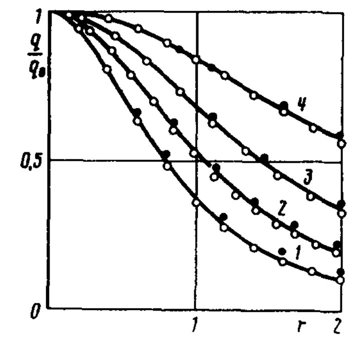

FIG. 2.

order accuracy of approximation in directions tangential to the surface [8]. The approximate and accurate values of the heat flux were compared for triaxial ellipsoids, elliptic paraboloids and hyperboloids; the ratio of the axes of the elliptic cross-section of the body was varied from 1 : 1 to 1 : 4. The comparison was made over a wide range of variation of the gas-dynamic flow parameters: $\gamma = 1.1$–1.667, $T_w = 0.01$–0.5 and $\mathrm{Re} = 10^2$–5×10^5. Some results of the comparison are shown in Figs 1–4. The dark and light points correspond to calculations using (1) and (10) respectively, while the lines in Figs 2–4 represent the numerical solution.

In Fig. 1 the analytical and numerical results were compared for the meridional section $\varphi = 45°$ of a paraboloid and a hyperboloid. Figure 2 shows the distributions of the relative heat flow along different meridional sections on the surface of an elliptic paraboloid with $k = 0.4$. Curves 1–4 correspond to the following values of φ: 0°, 45°, 63.4° and 90°. In Fig. 3 similar results are shown for an elliptic hyperboloid with a semi-aperture angle of 40° in the $y = 0$ plane and $k = 0.5$. The values of the gas-dynamic flow parameters when the numerical calculations were carried out were as follows: Fig. 2, $\mathrm{Re} = 10^3$, and in Fig. 3, $\mathrm{Re} = 10^4$; $T_w = 0.1$, $\gamma = 1.4$ and $\mathrm{Pr} = 0.71$.

The distributions of the heat along the surface of different ellipsoids $[x^2/a^2 + y^2/b^2 + (z/c - 1)^2 = 1]$ are shown in Fig. 4. Lines 1 and 2 correspond to the planes of symmetry $y = 0$ and $x = 0$ of the ellipsoids with a ratio of the axes of 3 : 2 : 1, while lines 3 and 4 are for an ellipsoid with a ratio of the axes of 0.7 : 0.3 : 1. The numerical calculations were carried out with $\mathrm{Re} = 5\times10^4$, $T_w = 0.15$ and $\gamma = 1.4$.

Apart from the good agreement between the analytical and numerical solutions, we can draw the following conclusions from an analysis of the results. For extremely oblate bodies, such as, an ellipsoid, when its transverse axis is much greater than its longitudinal axis, the heat flux begins to increase as one moves away from the stagnation point. For such oblate bodies, the maximum value of the heat flux is reached not at the stagnation point, but at a considerable distance from it, which can be explained (as in the case of flow around bodies at the angle of attack) by the considerable reduction in the radius of longitudinal curvature with distance from the stagnation point.

The above comparisons between the analytical and numerical solutions demonstrate that the above formulas, which express the relative heat flow on the surface of a three-dimensional body as a function of its geometrical characteristics, give completely satisfactory accuracy.

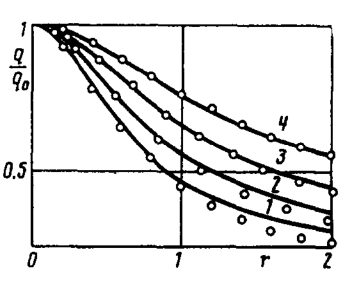

FIG. 3.

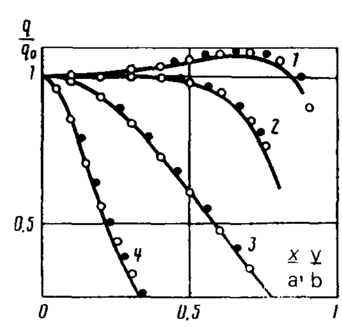

FIG. 4.

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FRACTAL CRACKS†

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An attempt is made to take into account the irregular structure of the surface of a real crack in describing the fracture process. The crack surface is modelled using a fractal set of fractional dimension. Using the self-similarity of the fractal, a hierarchical process of the transfer of elastic energy generated during the motion of the crack tip from one scale to another is suggested. Analysis of this process makes it possible to obtain asymptotic expressions for describing the behaviour of the cracks and displacements near the crack tip. It is shown that the fractal geometry of the crack leads to a change in the singular behaviour of the stress fields at the crack tip, and to the appearance of an anomalous dimensionally dependent factor in the expression for the stress intensity factor. Similar results are also obtained for branching fractal cracks. The propagation of a fractal crack in a brittle material is analysed from the positions of the Griffith's criterion.

THE surface of the fracture or crack formed as a result of the failure of most real materials is very irregular and is characterized by the presence of irregularities (peaks, hollows, serrations, etc.) of various different sizes. Therefore, a real crack hardly resembles, within the intermediate scale, ideal cracks with their smooth surfaces, which are usually considered in the theory of fracture. It is clear that the complex structure of the fracture surface which makes a significant contribution to the

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